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NON-LINEAR EQUATIONS FOR A SHALLOW UNSYMMETRICAL SANDWICH SHELL OF DOUBLE CURVATURE

by

ROBERT E. FULTON

A Technical Report
of a Research Program
Sponsored by
THE OFFICE OF NAVAL RESEARCH
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NON-LINEAR EQUATIONS FOR A SHALLOW UNSYMMETRICAL SANDWICH

SHELL OF DOUBLE CURVATURE

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Abstract

Equations are derived which governed the behavior of an elastic unsymmetrical doubly-curved sandwich shell. The face sheets may be of unequal thicknesses and of different materials. The equations include the non-linear effects; however, the restriction is made that the radii of curvature of the element are large when compared with the overall thickness of the sandwich.

The variational procedure is used to obtain three equations which govern the behavior of the shell and to determine the required boundary conditions. These resulting equations can be expressed in terms of a stress function, the radial deflection and a function which includes the contribution of the core. For the symmetrical case where the face sheets are of equal thicknesses and the same materials these equations are shown to reduce to those given by Grigolyuk in 1957. For an example the equations are used to obtain the critical load and the snap-through load of a square cylindrical shell section loaded in the longitudinal direction.

1. Introduction

The field of sandwich construction, while not new, has become quite important in recent years due to improvements in manufacturing techniques. It has long been recognized as an efficient method of obtaining a light weight compression member but the cost of construction prohibits its use. However, as new manufacturing methods are being developed which make the use of sandwiches economically feasible, the desirability to have more research data is becoming increasingly important.

The first significant contribution to an understanding of the behavior of sandwich shells was presented by Reissner (1)* wherein he evaluated the effects of shear deformations and core compressions which differentiate the sandwich theory from the ordinary shell theory based on the Kirchhoff-Love assumption. Since that time numerous papers have been published evaluating the effect of various parameters and discussing the analytical and experimental results of studies dealing with statically loaded cylindrical shells (2, 3, 4, 5, 6, 7, 8, 9, 10, 24).

More recent investigations have extended the theory to include doubly-curved shells (11, 12, 13), fully plastic cores (14), creep (15), minimum weight (16, 17, 18, 23), and free vibrations (19, 25). A recent major addition to the literature has been the accumulation of

* Numbers in parentheses refer to references at the end of the paper.

some of the significant results into two monographs edited by Aleksandrova (19). A rather thorough bibliography of the field and a discussion of some significant contributions is presented in a report by the author (17).

In this paper are developed non-linear equations governing the behavior of an elastic doubly-curved shallow sandwich shell with unsymmetrical faces. It is assumed that the core undergoes only transverse shear deformations and that a line through the undeformed core remains straight when deformed but not necessarily perpendicular to the middle surface of the shell. It is further assumed that the total thickness of the shell element is small compared to the radii of curvature. The face sheets, however, are assumed to satisfy the Kirchhoff-Love assumption and their thicknesses while not equal are small compared with the overall thickness of the sandwich section. It is likewise assumed that the core compression in a direction normal to the middle surface of the shell is negligible. The properties of each layer are different; however, for simplicity Poisson's ratio is assumed to be the same for all layers.

2. Compatibility Relationship for Each Face Sheet

If the expressions for strains for the ith face sheet in the x and y directions are noted as ϵ_{1i} and ϵ_{2i}, respectively, the transverse shear strain as γ_i, curvature in the x and y directions as χ_1 and χ_2, and the twist as χ_{12}, Equations (1) hold true for each of the separate face sheets.

$$\epsilon_{1i} = u_{ix} - \frac{w}{R_1} + \frac{w_x^2}{2} \qquad i = 1,2$$

$$\epsilon_{2i} = v_{iy} - \frac{w}{R_2} + \frac{w_y^2}{2}$$

$$\gamma_i = u_{iy} + v_{ix} + w_x w_y \tag{1}$$

$$\chi_1 = w_{xx}$$

$$\chi_2 = w_{yy}$$

$$\chi_{12} = w_{xy}$$

$$w_1 = w_2 = w$$

where u_i, v_i, w are the middle surface displacements of the ith face sheet considered in the x,y,z directions, respectively, and R_1 and R_2 are the radii of curvature of the plate elements in the x and y direction, respectively (see Figure 1). The subscripts x and y denote differentiation with respect to x and y, respectively.

By differentiating ϵ_{1i} twice with respect to y, ϵ_{2i} twice with respect to x, γ_i with respect to x and y and adding one obtains the compatibility relationship between the strains given as Equation (2).

$$\epsilon_{1iyy} + \epsilon_{2ixx} - \gamma_{ixy} + \frac{w_{xx}}{R_2} + \frac{w_{yy}}{R_1} + w_{xx}w_{yy} - w_{xy}^2 = 0 \qquad (2)$$

$$i = 1,2$$

3. Equations of Equilibrium

The first variation of the strain energy δV_i for the ith face sheet can be written as Equation (3) (see Figure 1).

$$\delta V_i = \int_{a_1}^{a_2}\int_{b_1}^{b_2} \Big[N_{1i}\delta\epsilon_{1i} + N_{2i}\,\delta\epsilon_{2i} + T_i\,\delta\gamma_i - M_{1i}\,\delta X_1$$

$$- M_{2i}\,\delta X_2 - 2H_i\,\delta X_{12} - p_i\delta w\Big]\,dy\,dx - \int_{b_1}^{b_2}[N_{1i}^{*}\,\delta u_i + T_{1i}^{*}\,\delta v_i$$

$$- M_{1i}^{*}\,\delta w_x + Q_{1i}^{*}\,\delta w]_{a_1}^{a_2}\;dy - \int_{a_1}^{a_2}[N_{2i}^{*}\,\delta v_i + T_{2i}^{*}\,\delta u_i -$$

$$M_{2i}^{*}\,\delta w_y + Q_{2i}^{*}\,\delta w]_{b_1}^{b_2}\,dx \qquad (3)$$

where N_{1i}, N_{2i}, and $T_{1i} = T_{2i} = T_i$ are the normal and shearing forces; M_{1i}, M_{2i}, and $H_{1i} = H_{2i} = H_i$ are the bending and twisting moments; and p_i is the external distributed load acting normal to the middle surface of the sheet.

The starred terms refer to the external boundary forces and the a's and b's are the coordinates of the edges of the shell in the x and y directions, respectively.

Let the moment-curvature and stress-strain relations for each face sheet be given by Equations (4).

$$M_{1i} = -D_i\,(w_{xx} + \mu\, w_{yy})$$

$$M_{2i} = -D_i\,(w_{yy} + \mu\, w_{xx}) \qquad (4)$$

$$H_i = -(1-\mu) D_i \; w_{xy} \tag{4}$$

$$N_{1i} = B_i \; (\epsilon_{1i} + \mu \; \epsilon_{2i})$$

$$N_{2i} = B_i \; (\epsilon_{2i} + \mu \; \epsilon_{1i})$$

$$T_i = G_i \; t_i \; \gamma_i = 1/2 \; (1-\mu) \; B_i \gamma_i$$

where

$$D_i = \frac{E_i t_i^3}{12(1-\mu^2)} \qquad B_i = \frac{E_i t_i}{(1-\mu^2)}$$

and where E_i, μ, t_i refer to Young's modulus, Poisson's ratio, and the thickness of the ith face sheet considered.

A comment should be made at this point regarding the notation used throughout the paper. Where forces and strains in the face sheets may differ both in direction and in the two face sheets, dual subscripts are used. When this occurs, the first subscript refers to the direction of the force of strain and the second refers to the face sheet under consideration. Thus ϵ_{21} signifies the strain in the y-direction in the upper face and M_{12} refers to the moment in the x-direction on the lower face.

The first variation of the strains in Equations (1) yields Equations (5).

$$\delta\epsilon_{1i} = \delta u_{ix} - \frac{\delta w}{R_1} + w_x \; \delta w_x$$

$$\delta\epsilon_{2i} = \delta v_{iy} - \frac{\delta w}{R_2} + w_y \; \delta w_y$$

$$\delta\gamma_i = \delta u_{iy} + \delta v_{ix} + w_x \; \delta w_y + w_y \; \delta w_x \tag{5}$$

$$\delta\chi_1 = \delta w_{xx}$$

$$\delta\chi_2 = \delta w_{yy}$$

$$\delta\chi_{12} = \delta w_{xy}$$

Substituting Equations (5) into Equation (3) and integrating by parts yields the first variation of the strain energy of the ith face sheet given by Equation (6)

$$\delta V_i = -\int_{a_1}^{a_2} \int_{b_1}^{b_2} \Big\{ [N_{1ix} + T_{iy}] \; \delta u_i + [N_{2iy} + T_{ix}] \; \delta v_i$$

$$+ [\frac{N_{1i}}{R_1} + (N_{1i}\, w_x)_x + \frac{N_{2i}}{R_2} + (N_{2i}\, w_y)_y + (T_i\, w_x)_y + (T_i\, w_y)_x$$

$$+ M_{1ixx} + M_{2iyy} + 2H_{ixy} + p_i]\ \delta w \Big\}\ dy\ dx \qquad (6)$$

$$- \int_{a_1}^{a_2} [(T_{2i}^{*} - T_i)\ \delta u_i + (N_{2i}^{*} - N_{2i})\ \delta v_i - (M_{2i}^{*} - M_{2i})\ \delta w_y$$

$$+ (Q_{2i}^{*} - N_{2i}\, w_y - T_i\, w_x - 2H_{ix} - M_{2iy})\ \delta w\ ]_{b_1}^{b_2}\ dx$$

$$- \int_{b_1}^{b_2} [(N_{1i}^{*} - N_{1i})\ \delta u_i + (T_{1i}^{*} - T_i)\ \delta v_i - (M_{1i}^{*} - M_{1i})\ \delta w_x$$

$$+ (Q_{1i}^{*} - N_{1i}\, w_x - T_i\, w_y - M_{1ix} - 2H_{iy})\ \delta w]_{a_1}^{a_2}\ dy - [[(2H_i)\ \delta w]_{a_1}^{a_2}]_{b_1}^{b_2}$$

where i has the range 1, 2.

The variation in the total strain energy of the two face sheets is equal to the sum of the two variations or

$$\delta V_F = \delta V_1 + \delta V_2 \qquad (7)$$

where V_F refers to the total strain energy of the faces and V_1 and V_2 are the strain energies of the upper and lower face sheets, respectively.

For the variation of the total strain energy of the shell there remains only to include the contribution of the core.

Consider now the equilibrium of the core element. It is assumed that the core undergoes only shear deformations and further that a line initially straight before deformation remains straight in the deformed state, however, not necessarily perpendicular to the middle surface of the shell. Therefore, a diagram of the values of u, the core displacements in the x direction is given by Figure 2.

Referring to Figure 2 it is seen that the displacements of a point in the core in the x and y directions are given by Equations (8), if the location of the neutral axis is known.

$$u = u_1 - \frac{t_1}{2} w_x - (\frac{\lambda}{c} + \frac{z}{c})\ [u_1 - u_2 - \frac{w_x}{2}(t_1 + t_2)]$$

$$v = v_1 - \frac{t_1}{2} w_y - (\frac{\lambda}{c} + \frac{z}{c}) [v_1 - v_2 - \frac{w_y}{2} (t_1 + t_2)] \qquad (8)$$

Let the shearing strain of the core in the xz plane and yz plane be denoted γ_{13} and γ_{23}, respectively. Taking into consideration Equations (8) there results

$$\gamma_{13} = u_z + w_x = \frac{h}{c} (\alpha - w_x) \qquad (9)$$

$$\gamma_{23} = v_z + w_y = \frac{h}{c} (\beta - w_y)$$

where

$$h = 1/2 (t_1 + t_2) + c \qquad (10)$$

$$\alpha = \frac{1}{h} (u_1 - u_2)$$

$$\beta = \frac{1}{h} (v_1 - v_2)$$

and where the subscripts 1 and 2 in Equations (10) refer to the upper and lower face sheets.

The shearing stresses τ_{13} and τ_{23} in the core are related to the strains by

$$\tau_{13} = G_c \gamma_{13}, \quad \tau_{23} = G_c \gamma_{23} \qquad (11)$$

where G_c is the shear modulus of the core.

Since the shearing strain is considered uniform across the core the first variation of the strain energy of the core δV_c is

$$\delta V_c = c \int_{a_1}^{a_2} \int_{b_1}^{b_2} (\tau_{13} \delta\gamma_{13} + \tau_{23} \delta\gamma_{23}) \, dy \, dx \qquad (12)$$

Using the relationships of Equations (9) and (11) and integrating by parts, one obtains from Equation (12)

$$\delta V_c = \frac{G_c h^2}{c} \int_{a_1}^{a_2} \int_{b_1}^{b_2} [(\alpha - w_x) \delta\alpha + (\beta - w_y) \delta\beta + (\alpha_x + \beta_y - \nabla^2 w) \delta w] \, dydx$$

$$- \int_{a_1}^{a_2} (\beta - w_y) \delta w \Big|_{b_1}^{b_2} dx - \int_{b_1}^{b_2} (\alpha - w_x) \delta w \Big|_{a_1}^{a_2} dy \qquad (13)$$

where ∇^2 is the Laplacian operator.

The total strain energy V of the shell is the sum of the contributions of the two face sheets and the core or

$$V = V_1 + V_2 + V_c \tag{14}$$

Taking first variation of Equation (14) yields

$$\delta V = \delta V_1 + \delta V_2 + \delta V_c = \delta V_F + \delta V_c \tag{15}$$

The expressions for δV_1 and δV_2 are given by Equation (6) by letting i equal 1 and 2. Substituting Equations (6) and (13) into Equation (15) yields

$$\delta V = - \int_{a_1}^{a_2}\int_{b_1}^{b_2} \Big\{ [N_{11x} + T_{1y}]\ \delta u_1 + [N_{12x} + T_{2y}]\ \delta u_2 + [N_{21y} + T_{1x}]\ \delta v_1$$

$$+ [N_{22y} + T_{2x}]\ \delta v_2 + [\frac{N_1}{R_1} + (N_1 w_x)_x + \frac{N_2}{R_2} + (N_2 w_y)_y + (Tw_x)_y + (Tw_y)_x$$

$$+ M_{1xx} + M_{2yy} + 2H_{xy} + p]\ \delta w \Big\}\ dy\ dx + \frac{Gh^2}{c} \int_{a_1}^{a_2}\int_{b_1}^{b_2} (\alpha - w_x)\ \delta\alpha$$

$$+ (\beta - w_y)\ \delta\beta + (\alpha_x + \beta_y - \nabla^2 w)\ \delta w]\ dy\ dx - \int_{a_1}^{a_2} [T_1^* - T_1)\ \delta u_1 + (T_2^*$$

$$- T_2)\ \delta u_2 + (N_{21}^* - N_{21})\ \delta v_1 + (N_{22}^* - N_{22})\ \delta v_2 - (M_2^* - M_2)\ \delta w_y + (Q_2^*$$

$$- N_2 w_y - Tw_x - 2H_x - M_{2y})\ \delta w]_{b_1}^{b_2}\ dx - \int_{b_1}^{b_2} [N_{11}^* - N_{11})\ \delta u_1 + (N_{12}^* - N_{12})\ \delta u_2$$

$$+ (T_1^* - T_1)\ \delta v_1 + (T_2^* - T_2)\ \delta v_2 - (M_1^* - M_1)\ \delta w_x + (Q_1^* - N_1 w_x - Tw_y$$

$$- M_{1x} - 2H_y) \, \delta w \Big]_{a_1}^{a_2} dy - [[(2H) \, \delta w]_{a_1}^{a_2}]_{b_1}^{b_2} - \frac{G_c h^2}{c} \int_{a_1}^{a_2} (\beta - v_y) \, \delta w \Big|_{b_1}^{b_2} dx$$

$$- \frac{G_c h^2}{c} \int_{b_1}^{b_2} (\alpha - v_x) \, \delta w \Big]_{a_1}^{a_2} dy \tag{16}$$

where $\delta\alpha = \frac{1}{h}(\delta u_1 - \delta u_2)$, $\delta\beta = \frac{1}{h}(\delta v_1 - \delta v_2)$

$$N_1 = N_{11} + N_{12} \qquad M_1 = M_{11} + M_{12} \qquad p = p_1 + p_2$$

$$N_2 = N_{21} + N_{22} \qquad M_2 = M_{21} + M_{22} \qquad Q_1 = Q_{11} + Q_{12}$$

$$T = T_1 + T_2 \qquad H = H_1 + H_2 \qquad Q_2 = Q_{21} + Q_{22}$$

and were the stars refer to values of the tractions at the boundary.

Introduce now the new variables, $\bar{u}$, $\bar{v}$ such that

$$\bar{u} = \frac{B_1 u_1 + B_2 u_2}{B_1 + B_2}$$
$$\bar{v} = \frac{B_1 v_1 + B_2 v_2}{B_1 + B_2} \tag{17}$$

These result in

$$u_1 = \bar{u} + \frac{B_2 h}{B_1 + B_2} \alpha \qquad v_1 = \bar{v} + \frac{B_2 h}{B_1 + B_2} \beta$$
$$u_2 = \bar{u} - \frac{B_1 h}{B_1 + B_2} \alpha \qquad v_2 = \bar{v} - \frac{B_1 h}{B_1 + B_2} \beta \tag{18}$$

and

$$\delta u_1 = \delta\bar{u} + \frac{B_2 h}{B_1 + B_2} \delta\alpha \qquad \delta v_1 = \delta\bar{v} + \frac{B_2 h}{B_1 + B_2} \delta\beta$$
$$\delta u_2 = \delta\bar{u} - \frac{B_1 h}{B_1 + B_2} \delta\alpha \qquad \delta v_2 = \delta\bar{v} - \frac{B_1 h}{B_1 + B_2} \delta\beta \tag{19}$$

Substitution of Equations (19) into Equation (16) yields

$$
\begin{aligned}
\delta V = &- \int_{a_1}^{a_2} \int_{b_1}^{b_2} \Big\{ (N_{1x} + T_y)\ \delta\bar{u} + \frac{h}{(B_1 + B_2)} \Big[(B_2 N_{11} - B_1 N_{12})_x + (B_2 T_1 \\
&- B_1 T_2)_y \Big] \delta\alpha + (N_{2y} + T_x)\ \delta\bar{v} + \frac{h}{(B_1 + B_2)} \Big[(B_2 N_{21} - B_1 N_{22})_y + (B_2 T_1 \\
&- B_1 T_2)_x \Big] \delta\beta + \Big[\frac{N_1}{R_1} + (N_1 w_x)_x + \frac{N_2}{R_2} + (N_2 w_y)_y + (T w_x)_y + (T w_y)_x + M_{1xx} \\
&+ M_{2yy} + 2H_{xy} + p \Big] \delta w + \frac{G_c h^2}{c} \Big[- (\alpha - w_x)\ \delta\alpha - (\beta - w_y)\ \delta\beta - (\alpha_x + \beta_y \\
&- \nabla^2 w)\ \delta w \Big] \Big\}\ dy\ dx - \int_{a_1}^{a_2} \Big\{ (T^* - T)\ \delta\bar{u} + \frac{h}{(B_1 + B_2)} \Big[(B_2 T_1^* - B_1 T_2^*) - (B_2 T_1 \\
&- B_1 T_2) \Big] \delta\alpha + (N_2^* - N_2)\ \delta\bar{v} + \frac{h}{(B_1 + B_2)} \Big[(B_2 N_{21}^* - B_1 N_{22}^*) - (B_2 N_{21} \\
&- B_1 N_{22}) \Big] \delta\beta - (M_2^* - M_2)\ \delta w_y + \Big[Q_2^* - N_2 w_y - T w_x - 2H_x - M_{2y} \\
&+ \frac{G_c h^2}{c} (\beta - w_y) \Big] \delta w \Big\}_{b_1}^{b_2} dx - \int_{b_1}^{b_2} \Big\{ (N_1^* - N_1)\ \delta\bar{u} + \frac{h}{(B_1 + B_2)} \Big[(B_2 N_{11}^* \\
&- B_1 N_{12}^*) - (B_2 N_{11} - B_1 N_{12}) \Big] \delta\alpha + (T^* - T)\ \delta\bar{v} + \frac{h}{(B_1 + B_2)} \Big[(B_2 T_1^* - B_1 T_2^*) \\
&- (B_2 T_1 - B_1 T_2) \Big] \delta\beta - (M_1^* - M_1)\ \delta w_x + \Big[Q_1^* - N_1 w_x - T w_y - M_{1x} - 2H_y \\
&+ \frac{G_c h^2}{c} (\alpha - w_x) \Big] \delta w \Big\}_{a_1}^{a_2} dy - \Big[\Big[(2H)\ \delta w \Big]_{a_1}^{a_2} \Big]_{b_1}^{b_2}
\end{aligned}
\qquad (20)
$$

From Equation (20) the equations of equilibrium are

$$N_{1x} + T_y = 0 \tag{21a}$$

$$N_{2y} + T_x = 0 \tag{21b}$$

$$\frac{h}{B_1 + B_2}\left[(B_2 N_{11} - B_1 N_{12})_x + (B_2 T_1 - B_1 T_2)_y\right] - \frac{G_c h^2}{c}(\alpha - w_x) = 0 \tag{21c}$$

$$\frac{h}{B_1 + B_2}\left[(B_2 N_{21} - B_1 N_{22})_y + (B_2 T_2 - B_1 T_2)_x\right] - \frac{G_c h^2}{c}(\beta - w_y) = 0 \tag{21d}$$

$$\frac{N_1}{R_1} + (N_1 w_x)_x + \frac{N_2}{R_2} + (N_2 w_y)_y + (Tw_x)_y + (Tw_y)_x + M_{1xx} + M_{2yy} + 2H_{xy} + p$$
$$- \frac{G_c h^2}{c}(\alpha_x + \beta_y - \nabla^2 w) = 0 \tag{21e}$$

4. Determination of Shell Equations

The equations of compatibility of the strains for each face sheet are given by Equation (2). By substituting the relation of Equations (4) into Equation (2) there results for the upper face

$$N_{11yy} - \mu N_{21yy} + N_{21xx} - \mu N_{11xx} - 2(1+\mu)T_{1xy}$$

$$+ B_1(1-\mu^2)\left(\frac{w_{xx}}{R_2} + \frac{w_{yy}}{R_1} + w_{yy}w_{xx} - w_{xy}^2\right) = 0 \tag{22}$$

and for the lower face

$$N_{12yy} - \mu N_{22yy} + N_{22xx} - \mu N_{12xx} - 2(1+\mu)T_{2xy}$$

$$+ B_2(1-\mu^2)\left(\frac{w_{xx}}{R_2} + \frac{w_{yy}}{R_1} + w_{yy}w_{xx} - w_{xy}^2\right) = 0 \tag{23}$$

where $B_i = \dfrac{E_i \quad t_i}{1 - \mu^2}$; $i = 1,2$

Adding Equations (22) and (23) yields

$$N_{1yy} - \mu N_{2yy} + N_{2xx} - \mu N_{1xx} - 2(1+\mu)T_{xy}$$

$$+ (B_1 + B_2)(1 - \mu^2)\left(\frac{w_{xx}}{R_2} + \frac{w_{yy}}{R_1} + w_{xx}w_{yy} - w_{xy}^2\right) = 0 \tag{24}$$

Equation (24) is the equation of compatibility for the two face sheets.

Consider now the equilibrium Equations (21a) and (21b). Introduce the Airy stress function F such that

$$N_1 = F_{yy}, \qquad N_2 = F_{xx}, \qquad T = -F_{xy} \tag{25}$$

Equations (25) satisfy the equilibrium Equations (21a) and (21b).

Substitution of Equations (25) into the compatibility Equation (24) yields

$$\nabla^4 F + (B_1 + B_2)(1-\mu^2)\left(\frac{w_{xx}}{R_2} + \frac{w_{yy}}{R_1} + w_{xx}\,w_{yy} - w_{xy}^{\,2}\right) = 0 \tag{26}$$

where $\nabla^4 = \nabla^2\nabla^2$.

Consider next the equilibrium Equations (21c) and (21d). If the relations of Equations (1), (4) and (10) are substituted into Equations (21c) and (21d) there results Equations (27a) and (27b).

$$\frac{(1-u)}{2}\nabla^2\alpha + \frac{(1+\mu)}{2}(\alpha_x + \beta_y)_x - \frac{G_c(B_1 + B_2)}{c\;B_1\,B_2}(\alpha - w_x) = 0 \tag{27a}$$

$$\frac{(1-\mu)}{2}\nabla^2\beta + \frac{(1+\mu)}{2}(\alpha_x + \beta_y)_y - \frac{G_c(B_1 + B_2)}{c\;B_1\,B_2}(\beta - w_y) = 0 \tag{27b}$$

Differentiating Equation (27a) with respect to x and Equation (27b) with respect to y and adding results in

$$\frac{c\;B_1\,B_2}{G_c(B_1 + B_2)}\nabla^2\varphi - \varphi + \nabla^2 w = 0 \tag{28}$$

where

$$\varphi = \alpha_x + \beta_y \tag{29}$$

Equation (28) may also be written as

$$\nabla^2 w = (1 - k\nabla^2)\,\varphi \tag{30}$$

where

$$k = \frac{c\;B_1\,B_2}{G_c(B_1 + B_2)} \tag{31}$$

Consider finally Equation (21e). Using the relations of Equations (4), (25) and (29) results in

$$(D_1 + D_2)\,\nabla^4 w - \left(\frac{1}{R_1} + w_{xx}\right)F_{yy} - \left(\frac{1}{R_2} + w_{yy}\right)F_{xx} + 2w_{xy}\,F_{xy}$$

$$+ \frac{G_c h^2}{c} (\varphi - \nabla^2 w) - p = 0 \qquad (32)$$

Thus the determination of the stresses and deformations of an unsymmetrical sandwich shell of double curvature has now been reduced to the solution of three non-linear partial differential equations with the appropriate boundary conditions. The three Equations (26), (30) and (32), are summarized below for convenience.

$$\nabla^4 F + (B_1 + B_2)(1-\mu^2)\left(\frac{w_{xx}}{R_2} + \frac{w_{yy}}{R_1} + w_{xx} w_{yy} - w_{xy}^2\right) = 0 \qquad (26)$$

$$\nabla^2 w = (1-k\nabla^2)\ \varphi \qquad (30)$$

$$(D_1+D_2)\ \nabla^4 w - \left(\frac{1}{R_1} + w_{xx}\right) F_{yy} - \left(\frac{1}{R_2} + w_{yy}\right) F_{xx} + 2w_{xy} F_{xy}$$

$$+ \frac{G_c h^2}{c} (\varphi - \nabla^2 w) - p = 0 \qquad (32)$$

where

$$\varphi = \alpha_x + \beta_y$$

$$k = \frac{c\ B_1 B_2}{G_c(B_1 + B_2)}$$

$$B_i = \frac{E_i t_i}{1-\mu^2}\ ,\qquad D_i = \frac{E_i t_i^{\,3}}{12(1-\mu^2)}$$

$$\alpha = \frac{1}{h}(u_1 - u_2),\quad \beta = \frac{1}{h}(v_1 - v_2)$$

$$N_1 = F_{yy},\quad N_2 = F_{xx},\quad T = -F_{xy}$$

It can be shown that Equations (26), (30) and (32) reduce to the equations given by Grigolyuk (12) if the face sheets are of equal thickness and of the same material. If $t_1 = t_2 = t$, and $E_1 = E_2 = E$, then $B_1 = B_2 = B$, $D_1 = D_2 = D$, $h = c + t$, $k = \frac{cB}{2G_c}$ and Equation (26) becomes

$$\nabla^4 F + 2B\ (1 - \mu^2)\left(\frac{w_{xx}}{R_2} + \frac{w_{yy}}{R_1} + w_{yy} w_{xx} - w_{xy}^2\right) = 0 \qquad (33)$$

Equations (30) and (32) then become Equations (34) and (35)

$$\nabla^2 w = (1 - k\nabla^2)\ \varphi \qquad (34)$$

$$2D\,\nabla^4 w - \left(\frac{1}{R_1} + w_{xx}\right) F_{yy} - \left(\frac{1}{R_2} + w_{yy}\right) F_{xx} + 2w_{xy} F_{xy} - p$$

$$+ \frac{G_c h^2}{c} (\varphi - \nabla^2 w) = 0 \tag{35}$$

where

$$\varphi = \alpha_x + \beta_y \tag{36}$$

As is shown by Grigolyuk φ can be eliminated from Equations (34) and (35) by proper differentiation to yield

$$2D\nabla^2\nabla^2\nabla^2 w - \frac{(c+t)^2}{c} G_c \left[\left(1 + \frac{2D}{D_o} \frac{c^2}{(c+t)^2}\right] \nabla^2\nabla^2 w\right. \tag{37}$$

$$+ \left(\nabla^2 - \frac{c\,G_c}{D_o}\right) \left[- \left(\frac{1}{R_1} + w_{xx}\right) F_{yy} - \left(\frac{1}{R_2} + w_{yy}\right) F_{xx} + 2F_{xy} w_{xy} - p \right] = 0$$

where $D_o = \frac{c^2 B}{2}$

Equation (37) and Equation (33) are of the same form as the results given by Reissner for a sandwich plate (21, 22) when $R_1 = R_2 = \infty$.

5. Boundary Conditions

From Equation (20) one can obtain the appropriate boundary conditions for the problem. Among the more important combinations of the boundary conditions which can be applied for a rectangular boundary are those given by Equations (38) through (49).

1. Along the edges $x = a_1$, $x = a_2$

Either $$N_1^* = F_{yy} \tag{38a}$$

or $$\bar{u} = 0 \tag{38b}$$

Either $$T^* = -F_{xy} \tag{39a}$$

or $$\bar{v} = 0 \tag{39b}$$

Either $$B_2 N_{11}^* - B_1 N_{12}^* = B_1 B_2\, h(\alpha_x + \mu\beta_y) \tag{40a}$$

or $$\alpha = 0 \tag{40b}$$

Either $$B_2T_1^* - B_1T_2^* = B_1B_2\, h\left(\frac{1-\mu}{2}\right)(\alpha_y + \beta_x) \qquad (41a)$$

or $$\beta = 0 \qquad (41b)$$

Either $$M_1^* = -(D_1 + D_2)(w_{xx} + \mu\, w_{yy}) \qquad (42a)$$

or $$w_y = 0 \qquad (42b)$$

Either $$\begin{aligned} Q_1^* = {} & F_{yy}w_x - F_{xy}w_y - 2(1-\mu)(D_1+D_2)w_{xyy} \\ & - (D_1+D_2)(w_{xxx} + \mu\, w_{xyy}) - \frac{G_c h^2}{c}(\alpha - w_x) \end{aligned} \qquad (43a)$$

or $$w = 0 \qquad (43b)$$

2. Along the edges $y = b_1$, $y = b_2$

Either $$T^* = -F_{xy} \qquad (44a)$$

or $$\bar{u} = 0 \qquad (44b)$$

Either $$N_2^* = F_{xx} \qquad (45a)$$

or $$\bar{v} = 0 \qquad (45b)$$

Either $$B_2T_1^* - B_1T_2^* = B_1B_2\, h\,\frac{(1-\mu)}{2}(\alpha_y + \beta_x) \qquad (46a)$$

or $$\alpha = 0 \qquad (46b)$$

Either $$B_2N_{21}^* - B_1N_{22}^* = B_1B_2\, h(\beta_y + \mu\alpha_x) \qquad (47a)$$

or $$\beta = 0 \qquad (47b)$$

Either $$M_2^* = -(D_1 + D_2)(w_{yy} + \mu\, w_{xx}) \qquad (48a)$$

or $$w_y = 0 \qquad (48b)$$

Either $$Q_2^* = F_{xx}w_y - F_{xy}w_x - 2(1-\mu)(D_1+D_2)w_{xxy}$$

$$- (D_1+D_2)(w_{yyy} + \mu\, w_{xxy}) - \frac{G_c h^2}{c}(\beta - w_y) \qquad (49a)$$

or $$w = 0 \qquad (49b)$$

6. Example

As an example of a solution to the aforesaid equations and boundary conditions, consider a square rectangular simply supported curved plate subjected to a normal force N^* parallel to its directrix along the edge $x = 0$, $x = a$ (Figure 3). It is required to determine the critical load for this problem and to investigate the post buckling behavior of the shallow shell.

For this problem $R_1 = \infty$, $R_2 = R$, $a = b$ and Equations (26), (30) and (32) become

$$\nabla^4 F = (B_1 + B_2)(1 - \mu^2)(w_{xy}^2 - w_{xx}w_{yy} - \frac{w_{xx}}{R}) \qquad (50)$$

$$(1 - k\nabla^2)\,\varphi = \nabla^2 w \qquad (51)$$

$$(D_1 + D_2)\nabla^4 w - w_{xx} F_{yy} - (\frac{1}{R} + w_{yy})F_{xx} + 2\, w_{xy} F_{xy} + \frac{G_c h^2}{c}(\varphi - \nabla^2 w) = 0 \qquad (52)$$

It is convenient to rewrite Equation (52) with the use of Equation (51) as

$$(D_1 + D_2)\nabla^4 w - w_{xx} F_{yy} - (\frac{1}{R} + w_{yy})F_{xx} + 2\, w_{xy} F_{xy} + \bar{k}\nabla^2\varphi = 0 \qquad (53)$$

where $$\bar{k} = \frac{h^2 B_1 B_2}{B_1 + B_2} \qquad (54)$$

and D_1, D_2, B_1, B_2 and k are as defined previously.

The boundary conditions for the problem are as follows:

1. along $x = 0$, $x = a$

$$F_{yy} = -N^* \qquad (55a)$$

$$F_{xy} = 0 \qquad (55b)$$

$$\alpha_x + \beta_y = \varphi = 0 \qquad (55c)$$

$$M_1 = 0 \tag{55d}$$

$$w = 0 \tag{55e}$$

2. along $y = 0$, $y = a$

$$F_{xy} = 0 \tag{56a}$$

$$F_{xx} = 0 \tag{56b}$$

$$\alpha_x + \beta_y = \varphi = 0 \tag{56c}$$

$$M_2^* = 0 \tag{56d}$$

$$w = 0 \tag{56e}$$

The method of solution will be that suggested by Kurshin in the collection of Aleksandrova (20) and is quite convenient for equations similar to the above.

Assume that
$$w = w_o \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} \tag{57}$$

where w_o is a constant.

When Equation (57) is substituted in Equation (50) and (51), the two resulting equations can be integrated exactly. Using these solutions Equation (52) will then be solved approximately using the method of Galerkin.

Substitution of Equation (57) into Equation (50) and integrating yields

$$F = \frac{C_1 w_o^2}{32} \left(\cos \frac{2\pi x}{a} + \cos \frac{2\pi y}{a}\right) + \frac{C_1 w_o}{4R} \frac{a^2}{\pi^2} \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} - \frac{N^* y^2}{2} \tag{58}$$

where
$$C_1 = (1 - \mu^2)(B_1 + B_2) \tag{59}$$

Likewise, substituting Equation (57) into Equation (51) and integrating yields

$$\varphi = - \frac{2\pi^2}{a^2} \frac{w_o}{\left(1 + \frac{2\pi^2}{a^2} k\right)} \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} \tag{60}$$

The solutions given as Equations (57), (58), and(60) satisfy only partially the boundary condition Equations (55) and (56). Equations (55c), (55d), (55e), (56c), (56d) and (56e) are fulfilled completely; however, Equations (55a), (55b), (56a), and (56b) are satisfied only on the average.

Equation (52) and the boundary condition Equations (55a), (55b), (56a) and (56b) can be replaced by the corresponding integral expressions from Equation (20). These are given as Equation (61).

$$\int_0^a \int_0^a \Big[(D_1 + D_2)\nabla^4 w - w_{xx}F_{yy} - (\tfrac{1}{R} + w_{yy})\, F_{xx}$$

$$+ 2\, w_{xy}F_{xy} + \bar{k}\,\nabla^2 \varphi\Big]\, \delta w\, dx dy + \int_0^a \Big[(F_{yy} + N^*)\ \delta\bar{u}\Big]_0^a dy \qquad (61)$$

$$- \int_0^a \Big[F_{xy}\ \delta\bar{v}\Big]_0^a dy + \int_0^a \Big[F_{xx}\ \delta\bar{v}\Big]_0^a dx - \int_0^a \Big[F_{xy}\ \delta\bar{u}\Big]_0^a dx = 0$$

With the exception of the displacements $\bar{u}$ and $\bar{v}$, everything is now known in Equation (61) in terms of w_o. To determine $\bar{u}$, $\bar{v}$ consider Equations (1) and (4). Equations (1) and (4) may be used to obtain

$$N_{1i} = B_i \left[u_{ix} - \frac{w}{R_1} + \frac{w_x^2}{2} + \mu(v_{iy} - \frac{w}{R_2} + \frac{w_y^2}{2})\right] \qquad (62a)$$

$$N_{2i} = B_i \left[v_{iy} - \frac{w}{R_2} + \frac{w_y^2}{2} + \mu(u_{ix} - \frac{w}{R_1} + \frac{w_x^2}{2})\right] \qquad (62b)$$

$$T_i = B_i \frac{(1 - \mu)}{2} \left[u_{iy} + v_{ix} + w_x w_y\right] \qquad (62c)$$

$$i = 1,\ 2$$

As defined previously

$$\bar{u} = \frac{B_1 u_1 + B_2 u_2}{B_1 + B_2}, \quad \bar{v} = \frac{B_1 v_1 + B_2 v_2}{B_1 + B_2} \qquad (63)$$

and

$$N_{11} + N_{12} = N_1 = F_{yy} \qquad (64a)$$

$$N_{21} = N_{22} = N_2 = F_{xx} \qquad (64b)$$

$$T_1 + T_2 = T = -F_{xy} \qquad (64c)$$

Substituting Equations (62) into Equation (64) and solving for $\bar{u}$ and $\bar{v}$ yields

$$\bar{u}_x = \frac{F_{yy} - \mu F_{xx}}{(1 - \mu^2)(B_1 + B_2)} + \left(\frac{w}{R_1} - \frac{w_x^2}{2}\right) \tag{65a}$$

$$\bar{v}_y = \frac{F_{yy} - \mu F_{yy}}{(1 - \mu^2)(B_1 + B_2)} + \left(\frac{w}{R_2} - \frac{w_y^2}{2}\right) \tag{65b}$$

$$\bar{u}_y + \bar{v}_x = \frac{-2 F_{xy}}{(1 - \mu)(B_1 + B_2)} - w_x w_y \tag{65c}$$

Letting $R_1 = \infty$, $R_2 = R$ and substituting the results of Equation (57) and (58), Equations (65) may be integrated to give for the problem under consideration

$$\bar{u} = - \frac{w_o^2}{8} \frac{\pi^2}{a^2} x - \frac{w_o^2 \pi}{16a} \sin \frac{2\pi x}{a} \left(2 \sin^2 \frac{\pi y}{a} - \mu\right)$$

$$+ \frac{(1 - \mu)}{4} \frac{w_o}{R} \frac{a}{\pi} \cos \frac{\pi x}{a} \sin \frac{\pi y}{a} - \frac{N x^*}{C_1} \tag{66a}$$

$$\bar{v} = - \frac{w_o^2}{8} \frac{\pi^2}{a^2} y - \frac{w_o^2 \pi}{16a} \sin \frac{2\pi y}{a} \left(2 \sin^2 \frac{\pi x}{a} - \mu\right)$$

$$+ \frac{(-3-\mu)}{4} \frac{w_o}{R} \frac{a}{\pi} \sin \frac{\pi x}{a} \cos \frac{\pi y}{a} + \mu \frac{N y^*}{C_1} \tag{66b}$$

Using the results of Equations (66), (57), (58) and (60) in the integral equation (61) and carrying out the integration yields

$$- \frac{\pi^2}{4} N^* w_o + \frac{C_1}{32} \frac{\pi^4}{a^2} w_o^3 + \frac{C_1 w_o^2}{4R} (-1+\mu)$$

$$+ w_o \left[(D_1 + D_2) \frac{\pi^4}{a^2} + \frac{C_1}{16} \frac{a^2}{R^2} + \frac{\bar{k} \pi^4}{a^2\left(1 + \frac{2\pi^2}{a^2} k\right)} \right] = 0 \tag{67}$$

Solving for N^* there results

$$N^* = 4(D_1 + D_2)\frac{\pi^2}{a^2} + \frac{C_1}{4\pi^2}\frac{a^2}{R^2} + \frac{4\bar{k}\,\pi^2}{a^2\left(1 + \frac{2\pi^2}{a^2}k\right)}$$

$$- \frac{C_1(1-\mu)}{\pi^2 R}\,w_o + \frac{C_1\pi^2}{8\,a^2}\,w_o^2 \qquad (68)$$

Dropping the non-linear terms the critical load is obtained

$$N_u^* = 4(D_1 + D_2)\frac{\pi^2}{a^2} + \frac{C_1}{4\pi^2}\frac{a^2}{R^2} + \frac{4\bar{k}\,\pi^2}{a^2\left(1 + \frac{2\pi^2 k}{a^2}\right)} \qquad (69)$$

where k, $\bar{k}$, and C_1 are given by Equations (31), (54) and (59).

Equation (69) gives the upper critical load just as the shell snaps through. The lower critical load which corresponds to the snap-through condition may be obtained by considering the non-linear terms.

Differentiating Equation (68) with respect to w_o and solving for w_o yields

$$w_o = \frac{4a^2}{\pi^4 R}(1-\mu) \qquad (70)$$

Substituting Equation (70) into Equation (68) gives the lower value of the critical load N_L^* which results after loss of stability.

$$N_L^* = N_u^* - \frac{2C_1(1-\mu)^2\,a^2}{\pi^6\;R^2} \qquad (71)$$

A measure of the energy loss resulting from shell buckling may be obtained by investigating the ratio of the upper and lower critical loads for the various parameters of the shell.

$$\frac{N_u^*}{N_L^*} = \frac{N_u^*}{N_u^* - \frac{2C_1}{\pi^6}(1-\mu)^2\frac{a^2}{R^2}} \qquad (72)$$

or

$$\frac{N_u^*}{N_L^*} = \frac{1}{1-\epsilon} \qquad (73)$$

where

$$\epsilon = \frac{\frac{2C_1}{\pi^6}(1-\mu)^2 a^2/R^2}{4(D_1 + D_2)\pi^2/a^2 + \frac{C_1}{4\pi^2} a^2/R^2 + \frac{4\bar{k}\,\pi^2}{a^2\left(1 + \frac{2\pi^2 k}{a^2}\right)}} \tag{74}$$

A maximum value of ϵ may be determined by considering only the middle term in the denominator of Equation (74).

$$\epsilon = \frac{8}{\pi^4}(1-\mu)^2 = .046, \quad \text{for } \mu = 0.25$$

and

$$\frac{N_u^*}{N_L^*} = 1.05$$

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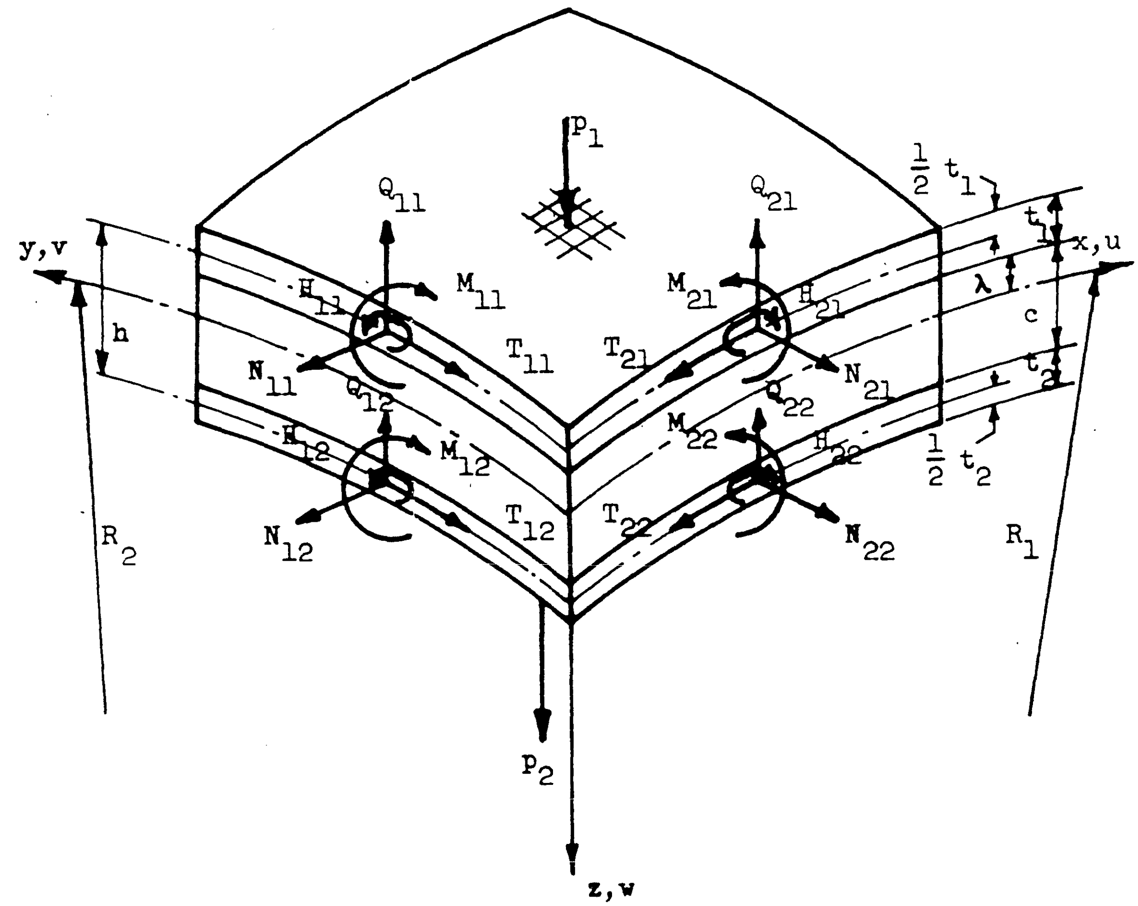

FIGURE 1 FORCES ON THE SANDWICH SHELL ELEMENT

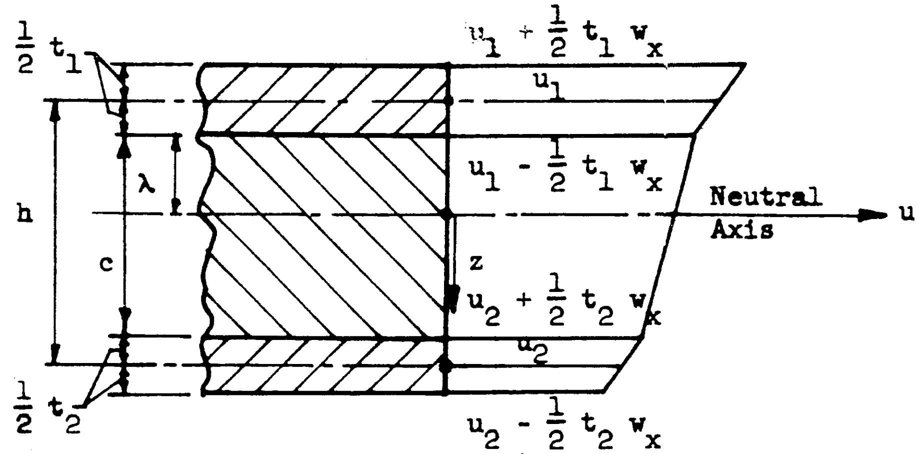

FIGURE 2 PLOT OF u - DISPLACEMENTS THROUGH THE THICKNESS OF THE SHELL

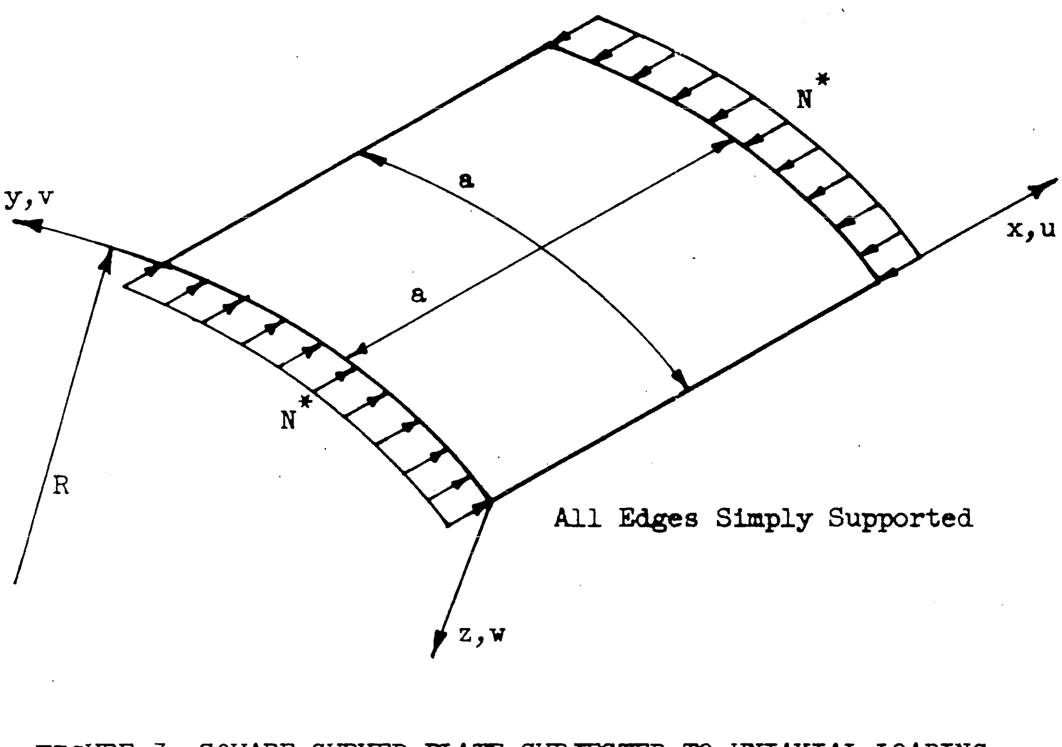

FIGURE 3 SQUARE CURVED PLATE SUBJECTED TO UNIAXIAL LOADING